

LINEAR FORMS AND COMPLEMENTING SETS OF INTEGERS

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ABSTRACT. Let $\varphi(x_1, \dots, x_h, y) = u_1x_1 + \dots + u_hx_h + vy$ be a linear form with nonzero integer coefficients u_1, \dots, u_h, v . Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of finite sets of integers and let B be an infinite set of integers. Define the representation function associated to the form φ and the sets \mathcal{A} and B as follows:

$$R_{\mathcal{A},B}^{(\varphi)}(n) = \text{card}(\{(a_1, \dots, a_h, b) \in A_1 \times \dots \times A_h \times B : \varphi(a_1, \dots, a_h, b) = n\}).$$

If this representation function is constant, then the set B is periodic and the period of B will be bounded in terms of the diameter of the finite set $\{\varphi(a_1, \dots, a_h, 0) : (a_1, \dots, a_h) \in A_1 \times \dots \times A_h\}$.

1. REPRESENTATION FUNCTIONS FOR LINEAR FORMS

Let $h \geq 1$ and let

$$\psi(x_1, \dots, x_h) = u_1x_1 + \dots + u_hx_h$$

be a linear form with nonzero integer coefficients u_1, \dots, u_h . Let

$$\mathcal{A} = (A_1, \dots, A_h)$$

be an h -tuple of sets of integers. The *image* of ψ with respect to \mathcal{A} is the set

$$\psi(\mathcal{A}) = \{\psi(a_1, \dots, a_h) : (a_1, \dots, a_h) \in A_1 \times \dots \times A_h\}.$$

Then $\psi(\mathcal{A}) \neq \emptyset$ if and only if $A_i \neq \emptyset$ for all $i = 1, \dots, h$. For $\psi(\mathcal{A}) \neq \emptyset$, we define the *diameter of \mathcal{A} with respect to ψ* by

$$D_{\mathcal{A}}^{(\psi)} = \text{diam}(\psi(\mathcal{A})) = \sup(\psi(\mathcal{A})) - \inf(\psi(\mathcal{A})).$$

We have $D_{\mathcal{A}}^{(\psi)} > 0$ if and only if $|A_i| > 1$ for some i .

For every integer n , we define the *representation function* associated to ψ by

$$R_{\mathcal{A}}^{(\psi)}(n) = \text{card}(\{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : \psi(a_1, \dots, a_h) = n\}).$$

Then $n \in \psi(\mathcal{A})$ if and only if $R_{\mathcal{A}}^{(\psi)}(n) > 0$.

Let $\ell \geq 1$ and let

$$\omega(y_1, \dots, y_\ell) = v_1y_1 + \dots + v_\ell y_\ell$$

be another linear form with nonzero integer coefficients v_1, \dots, v_ℓ . Consider the linear form

$$\varphi(x_1, \dots, x_h, y_1, \dots, y_\ell) = \psi(x_1, \dots, x_h) + \omega(y_1, \dots, y_\ell).$$

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Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of sets of integers and let $\mathcal{B} = (B_1, \dots, B_\ell)$ be an ℓ -tuple of sets of integers. The *image* of φ with respect to $(\mathcal{A}, \mathcal{B})$ is the set

$$\begin{aligned} \varphi(\mathcal{A}, \mathcal{B}) &= \psi(\mathcal{A}) + \omega(\mathcal{B}) \\ &= \{ \psi(a_1, \dots, a_h) + \omega(b_1, \dots, b_\ell) : (a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ &\quad \text{and } (b_1, \dots, b_\ell) \in B_1 \times \dots \times B_\ell \}. \end{aligned}$$

We define the *representation function associated to φ , \mathcal{A} , and \mathcal{B}* by

$$\begin{aligned} R_{\mathcal{A}, \mathcal{B}}^{(\varphi)}(n) &= \text{card}(\{(a_1, \dots, a_h, b_1, \dots, b_\ell) \in A_1 \times \dots \times A_h \times B_1 \times \dots \times B_\ell : \\ &\quad \varphi(a_1, \dots, a_h, b_1, \dots, b_\ell) = n\}). \end{aligned}$$

For every positive integer m , we define the *modular representation function associated to φ* by

$$\begin{aligned} R_{\mathcal{A}, \mathcal{B}; m}^{(\varphi)}(n) &= \text{card}(\{(a_1, \dots, a_h, b_1, \dots, b_\ell) \in A_1 \times \dots \times A_h \times B_1 \times \dots \times B_\ell : \\ &\quad \varphi(a_1, \dots, a_h, b_1, \dots, b_\ell) \equiv n \pmod{m}\}). \end{aligned}$$

If $\ell = 1$ and $\mathcal{B} = (B)$, then we write $\varphi(\mathcal{A}, \mathcal{B}) = \varphi(\mathcal{A}, B)$, $R_{\mathcal{A}, \mathcal{B}}^{(\varphi)}(n) = R_{\mathcal{A}, B}^{(\varphi)}(n)$, and $R_{\mathcal{A}, \mathcal{B}; m}^{(\varphi)}(n) = R_{\mathcal{A}, B; m}^{(\varphi)}(n)$.

Notation. Let \mathbf{Z} and \mathbf{N}_0 denote the set of integers and the set of nonnegative integers, respectively. We denote the cardinality of the set S by $|S|$ or by $\text{card}(S)$. We denote the integer part of the real number x by $[x]$.

2. COMPLEMENTING SETS

A classical problem in additive number theory is the study of complementing pairs of sets of integers, that is, pairs (A, B) such that every integer has a unique representation in the form $a + b$, with $a \in A$ and $b \in B$. This is the case $h = 1$, $\psi(x) = x$, $\omega(y) = y$, and $\varphi(x, y) = x + y$ of the general problem of representations of integers by linear forms. There are many beautiful results and open problems about complementing sets for the integers. For example, if A is a finite set of integers and if B is an infinite set of integers such that the pair (A, B) is complementing, then B must be a periodic set, that is, a union of congruence classes modulo m for some positive integer m (Newman [6]). There are also upper and lower bounds on the period m as a function of the diameter of the set A (Biro [1], Kolountzakis [3], Ruzsa [9, Appendix], Steinberger [8]). In general, it is known that every pair (A, B) of complementing sets with A finite must satisfy a certain cyclotomy condition, but it is an open problem to determine if a finite set A of integers has a complement.

Complementing pairs have also been studied for sets of lattice points (Hansen [2], Nathanson [4], Niven [7]). If (A, B) is a pair of sets of lattice points such that A is finite and every lattice point has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$, then it is an open problem to determine if the set B must be periodic.

The object of this paper is to begin the study of complementing sets of integers with respect to an arbitrary linear form $\varphi(x_1, \dots, x_h, y_1, \dots, y_\ell)$. Let \mathcal{A} be an h -tuple of sets of integers and \mathcal{B} an ℓ -tuple of sets of integers. The pair $(\mathcal{A}, \mathcal{B})$ is called *complementing* with respect to φ if $R_{\mathcal{A}, \mathcal{B}}^{(\varphi)}(n) = 1$ for all $n \in \mathbf{Z}$, that is, if every integer n has a unique representation in the form $n = \psi(a_1, \dots, a_h) + \omega(b_1, \dots, b_\ell)$, where $a_i \in A_i$ for $i = 1, \dots, h$ and $b_j \in B_j$ for $j = 1, \dots, \ell$. The pair $(\mathcal{A}, \mathcal{B})$ is

called *t-complementing* with respect to φ if $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n) = t$ for all $n \in \mathbf{Z}$. The pair $(\mathcal{A}, \mathcal{B})$ is called *t-complementing modulo m* with respect to φ if $R_{\mathcal{A},\mathcal{B};m}^{(\varphi)}(\ell) = t$ for all $\ell \in \{0, 1, \dots, m-1\}$.

The pair $(\mathcal{A}, \mathcal{B})$ is called *periodic with respect to φ* if the representation function $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}$ is periodic, that is, if there is a positive integer m such that $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n+m) = R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n)$ for all integers n . The pair $(\mathcal{A}, \mathcal{B})$ is called *eventually periodic with respect to φ* if the representation function $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}$ is eventually periodic, that is, if there exist integers $m \geq 1$ and n_0 such that $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n+m) = R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n)$ for all integers $n \geq n_0$.

We consider the case $\ell = 1$. Suppose that $\varphi(x_1, \dots, x_h, y) = \psi(x_1, \dots, x_h) + vy$ is a linear form with nonzero integer coefficients, and that \mathcal{A} is an h -tuple of finite sets of integers and B is a set of integers such that the pair (\mathcal{A}, B) is *t-complementing* with respect to φ . We shall prove that the set B is periodic, and obtain an upper bound for the period of B in terms of the diameter $D_{\mathcal{A}}^{\psi}$ of the finite set $\psi(\mathcal{A})$. We also obtain a cyclotomic condition related to *t-complementing* sets modulo m , and describe a compactness argument that allows us to solve an inverse problem related to representation functions associated with linear forms.

3. LINEAR FORMS AND PERIODICITY

Theorem 1. *Let $h \geq 1$ and let*

$$\varphi(x_1, \dots, x_h, y) = u_1x_1 + \dots + u_hx_h + vy$$

*be a linear form with nonzero integer coefficients u_1, \dots, u_h, v . Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of nonempty finite sets of integers, and let B be an infinite set of integers. If (\mathcal{A}, B) is *t-complementing* respect to φ , then B is periodic, that is, there is a positive integer m such that B is a union of congruence classes modulo m .*

Proof. If $v < 0$, then we replace φ with $-\varphi$. Thus, we can assume without loss of generality that $v \geq 1$.

If $|A_i| = 1$ for all $i = 1, \dots, h$, then the linear form φ represents all integers if and only if $v = 1$ and $B = \mathbf{Z}$, and the Theorem holds with $m = 1$. Thus, we can also assume that $|A_i| > 1$ for at least one i .

Consider the linear form

$$\psi(x_1, \dots, x_h) = u_1x_1 + \dots + u_hx_h.$$

We have

$$\varphi(a_1, \dots, a_h, b) = \psi(a_1, \dots, a_h) + vb$$

for all $(a_1, \dots, a_h) \in A_1 \times \dots \times A_h$ and $b \in B$. Let $g_{\min} = \min(\psi(A_1, \dots, A_h))$ and $g_{\max} = \max(\psi(A_1, \dots, A_h))$. Since $|A_i| > 1$ for some $i \in \{1, 2, \dots, h\}$, it follows that $g_{\min} < g_{\max}$ and

$$D_{\mathcal{A}}^{(\psi)} = \text{diam}(\psi(A_1, \dots, A_h)) = g_{\max} - g_{\min} \geq 1.$$

Let

$$G_{\min} = \{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : \psi(a_1, \dots, a_h) = g_{\min}\}$$

and

$$G_{\max} = \{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : \psi(a_1, \dots, a_h) = g_{\max}\}.$$

Then

$$|G_{\min}| = R_{\mathcal{A}}^{(\psi)}(g_{\min}) \geq 1$$

and

$$|G_{\max}| = R_{\mathcal{A}}^{(\psi)}(g_{\max}) \geq 1.$$

Let $\chi_B : \mathbf{R} \rightarrow \{0, 1\}$ denote the characteristic function of the set B , that is,

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

We have

$$\varphi(a_1, \dots, a_h, b) = \psi(a_1, \dots, a_h) + vb = n$$

if and only if

$$b = \frac{n - \psi(a_1, \dots, a_h)}{v} \in B.$$

It follows that

$$R_{\mathcal{A}}^{(\varphi)}(n) = \sum_{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h} \chi_B \left(\frac{n - \psi(a_1, \dots, a_h)}{v} \right)$$

for all $n \in \mathbf{Z}$. We can also write

$$\begin{aligned} R_{\mathcal{A}}^{(\varphi)}(n) &= \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(\frac{n - \psi(a_1, \dots, a_h)}{v} \right) \\ &\quad + |G_{\min}| \chi_B \left(\frac{n - g_{\min}}{v} \right). \end{aligned}$$

Replacing n by $vn + g_{\min}$, we obtain the identity

$$\begin{aligned} R_{\mathcal{A}}^{(\varphi)}(vn + g_{\min}) &= \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(\frac{vn + g_{\min} - \psi(a_1, \dots, a_h)}{v} \right) \\ &\quad + |G_{\min}| \chi_B(n). \end{aligned}$$

Equivalently,

$$\begin{aligned} |G_{\min}| \chi_B(n) &= R_{\mathcal{A}}^{(\varphi)}(vn + g_{\min}) \\ &\quad - \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(n - \frac{\psi(a_1, \dots, a_h) - g_{\min}}{v} \right). \end{aligned}$$

Since $g_{\min} < \psi(a_1, \dots, a_h) \leq g_{\max}$ for all h -tuples $(a_1, \dots, a_h) \notin G_{\min}$, it follows that

$$0 < \frac{1}{v} \leq \frac{\psi(a_1, \dots, a_h) - g_{\min}}{u_h} \leq \frac{g_{\max} - g_{\min}}{u_h}.$$

Similarly, replacing n by $vn + g_{\max}$, we obtain the identity

$$\begin{aligned} |G_{\max}| \chi_B(n) &= R_{\mathcal{A}}^{(\varphi)}(vn + g_{\max}) \\ &\quad - \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(n + \frac{g_{\max} - \psi(a_1, \dots, a_h)}{v} \right). \end{aligned}$$

Since $g_{\min} \leq \psi(a_1, \dots, a_h) < g_{\max}$ for $(a_1, \dots, a_h) \notin G_{\max}$, it follows that

$$0 < \frac{1}{v} \leq \frac{g_{\max} - \psi(a_1, \dots, a_h)}{v} \leq \frac{g_{\max} - g_{\min}}{v}.$$

We define the nonnegative integer

$$(1) \quad d = \left\lceil \frac{\text{diam}(\psi(A_1, \dots, A_h))}{v} \right\rceil = \left\lceil \frac{g_{\max} - g_{\min}}{v} \right\rceil.$$

If the pair (\mathcal{A}, B) is t -complementing with respect to φ , then $R_{\mathcal{A}, B}^{(\varphi)}(n) = t$ for all $n \in \mathbf{Z}$, and so

$$|G_{\min}| \chi_B(n) = t - \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(n - \frac{\psi(a_1, \dots, a_h) - g_{\min}}{v} \right)$$

and

$$|G_{\max}| \chi_B(n) = t - \sum_{\substack{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h \\ (a_1, \dots, a_h) \notin G_{\min}}} \chi_B \left(n + \frac{g_{\max} - \psi(a_1, \dots, a_h)}{v} \right)$$

These formulae allow us to compute the characteristic function χ_B recursively for all integers if we know the value of χ_B for any d consecutive integers.

Consider the d -tuple

$$\mathcal{B}(j) = (\chi_B(j), \chi_B(j+1), \dots, \chi_B(j+d-1)) \in \{0, 1\}^d.$$

Since there are only 2^d binary sequences of length d , it follows from the pigeonhole principle that there are integers j_1, j_2 such that $0 \leq j_1 < j_2 \leq 2^d$ and $\mathcal{B}(j_1) = \mathcal{B}(j_2)$. Let $m = j_2 - j_1$. Then

$$1 \leq m \leq 2^d$$

and $\chi_B(n) = \chi_B(n+m)$ for $n = j_1, \dots, j_1 + d - 1$. The recursion formulae imply that $\chi_B(n) = \chi_B(n+m)$ for all integers n . This completes the proof. \square

4. LINEAR FORMS AND CYCLOTOMY

Theorem 2. *Let $h \geq 1$ and let*

$$\psi(x_1, \dots, x_h, y) = u_1 x_1 + \dots + u_h x_h$$

be a linear form with nonzero integer coefficients u_1, \dots, u_h . Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of nonempty finite sets of integers. Consider the modular representation function

$$R_{\mathcal{A}, m}^{(\psi)}(n) = \text{card}(\{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : \psi(a_1, \dots, a_h) \equiv n \pmod{m}\}).$$

and the generating functions

$$F_{A_i}(z) = \sum_{a_i \in A_i} z^{a_i} \quad \text{for } i = 1, \dots, h.$$

For $m \geq 1$, define the polynomial

$$\Lambda_m(z) = 1 + z + z^2 + \dots + z^{m-1}.$$

The h -tuple \mathcal{A} is t -complementing modulo m with respect to ψ if and only if there exists a nonnegative integer L such that

$$(2) \quad z^L F_{A_1}(z^{u_1}) \dots F_{A_h}(z^{u_h}) \equiv t \Lambda_m(z) \pmod{z^m - 1}.$$

Proof. The generating functions $F_{A_i}(z)$ are nonzero Laurent polynomials for $i = 1, \dots, h$. The function

$$F(z) = F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h})$$

is also a nonzero Laurent polynomial with integer coefficients. Choose a nonnegative integer L such that $z^L F(z)$ is a polynomial.

The sets A_1, \dots, A_h are finite, and so $\psi(\mathcal{A})$ is finite. We have $R_{\mathcal{A}}^{(\psi)}(n) \geq 1$ if and only if $n \in \psi(\mathcal{A})$. For $\ell = 0, 1, \dots, m-1$, we consider the finite set

$$\mathcal{I}_\ell = \{i \in \mathbf{Z} : R_{\mathcal{A}}^{(\psi)}(\ell + im) \geq 1\}.$$

Since $F_{A_i}(z^{u_i}) = \sum_{a_i \in A_i} z^{u_i a_i}$ for $i = 1, \dots, h$, it follows that

$$\begin{aligned} F(z) &= F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h}) \\ &= \sum_{a_1 \in A_1} \cdots \sum_{a_h \in A_h} z^{u_1 a_1 + \cdots + u_h a_h} \\ &= \sum_{a_1 \in A_1} \cdots \sum_{a_h \in A_h} z^{\psi(a_1, \dots, a_h)} \\ &= \sum_{n \in \psi(\mathcal{A})} R_{\mathcal{A}}^{(\psi)}(n) z^n \\ &= \sum_{\ell=0}^{m-1} \sum_{\substack{n \in \psi(\mathcal{A}) \\ n \equiv \ell \pmod{m}}} R_{\mathcal{A}}^{(\psi)}(n) z^n \\ &= \sum_{\ell=0}^{m-1} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + im}. \end{aligned}$$

Since

$$z^L F(z) = \sum_{\ell=0}^{m-1} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + L + im}$$

is a polynomial, it follows that $\ell + L + im \geq 0$ for all $\ell \in \{0, 1, \dots, m-1\}$ and $i \in \mathcal{I}_\ell$. Applying the division algorithm for integers, we can write

$$\ell + L = \alpha(\ell) + \beta(\ell)m$$

where $0 \leq \alpha(\ell) \leq m-1$ for $\ell = 0, 1, \dots, m-1$. Moreover, if $\ell \not\equiv \ell' \pmod{m}$, then $\alpha(\ell) \neq \alpha(\ell')$ and so

$$\{\alpha(0), \alpha(1), \dots, \alpha(m-1)\} = \{0, 1, \dots, m-1\}.$$

Equivalently,

$$\sum_{\ell=0}^{m-1} z^{\alpha(\ell)} = \sum_{\ell=0}^{m-1} z^\ell = \Lambda_m(z).$$

If $i \in \mathcal{I}_\ell$, then the inequality

$$\ell + L + im = \alpha(\ell) + (\beta(\ell) + i)m \geq 0$$

implies that $\beta(\ell) + i \geq 0$. Therefore, for each $\ell \in \{0, 1, \dots, m-1\}$ there is a polynomial $p_\ell(z)$ with integral coefficients such that

$$\begin{aligned} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell+L+im} &= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell) + (\beta(\ell) + i)m} \\ &= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell)} (1 + (z^m - 1))^{\beta(\ell) + i} \\ &= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell)} + (z^m - 1) p_\ell(z) \\ &= R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)} + (z^m - 1) p_\ell(z). \end{aligned}$$

It follows that

$$\begin{aligned} z^L F(z) &= \sum_{\ell=0}^{m-1} \sum_{i \in \mathbf{Z}} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell+L+im} \\ &= \sum_{\ell=0}^{m-1} R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)} + (z^m - 1) \sum_{\ell=0}^{m-1} p_\ell(z) \\ &= r_L(z) + (z^m - 1) q_L(z), \end{aligned}$$

where

$$q_L(z) = \sum_{\ell=0}^{m-1} p_\ell(z)$$

and

$$r_L(z) = \sum_{\ell=0}^{m-1} R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)}$$

is a polynomial of degree at most $m-1$. By the division algorithm for polynomials, this representation of $z^L F(z)$ is unique.

Suppose that $\mathcal{A} = (A_1, \dots, A_h)$ is a t -complementing h -tuple modulo m . Then $R_{\mathcal{A},m}(\ell) = t$ for all ℓ , and

$$r_L(z) = \sum_{\ell=0}^{m-1} t z^{\alpha(\ell)} = t \Lambda_m(z).$$

Therefore,

$$z^L F(z) = t \Lambda_m(z) + (z^m - 1) q_L(z)$$

and condition (2) is satisfied.

Conversely, suppose that the generating functions $F_{A_1}(z), \dots, F_{A_h}(z)$ satisfy condition (2) for some nonnegative integer L . By the uniqueness of the polynomial division algorithm, we have

$$\sum_{\ell=0}^{m-1} t z^\ell = t \Lambda_m(z) = r_L(z) = \sum_{\ell=0}^{m-1} R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)}.$$

Since

$$\{\alpha(0), \alpha(1), \dots, \alpha(m-1)\} = \{0, 1, \dots, m-1\},$$

it follows that $R_{\mathcal{A},m}^{(\psi)}(\ell) = t$ for all $\ell \in \{0, 1, \dots, m-1\}$, and so $\mathcal{A} = (A_1, \dots, A_h)$ is a t -complementing h -tuple modulo m . This completes the proof. \square

5. AN INVERSE PROBLEM FOR LINEAR FORMS

There are several inverse problems for representation functions associated to linear forms. For example, let $\varphi(x_1, \dots, x_h, y)$ be a form in $h + 1$ variables and let $f : \mathbf{Z} \rightarrow \mathbf{N}_0 \cup \{\infty\}$ be a function. If $\mathcal{A} = (A_1, \dots, A_h)$ is an h -tuple of sets of integers, does there exist a set B such that the pair (\mathcal{A}, B) satisfies $R_{\mathcal{A}, B}^{(\varphi)}(n) = f(n)$ for all $n \in \mathbf{Z}$? In this section we use a compactness argument to obtain a result in the case that $\mathcal{A} = (A_1, \dots, A_h)$ is an h -tuple of finite sets.

Theorem 3. *Let $h \geq 1$ and let*

$$\varphi(x_1, \dots, x_h, y) = u_1 x_1 + \dots + u_h x_h + u_h x_h + v y$$

be a linear form with nonzero integer coefficients u_1, \dots, u_h, v . Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of nonempty finite sets of integers. Let $f : \mathbf{Z} \rightarrow \mathbf{N}_0 \cup \{\infty\}$ be a function. Suppose that there is a strictly increasing sequence $\{L_N\}_{N=1}^\infty$ of positive integers with the property that, for every $N \geq 1$, there exists a set B_N of integers that satisfies

$$R_{\mathcal{A}, B_N}^{(\varphi)}(n) = f(n) \quad \text{for } |n| \leq L_N.$$

Then there exists a set B such that

$$R_{\mathcal{A}, B}^{(\varphi)}(n) = f(n) \quad \text{for all } n \in \mathbf{Z}.$$

Proof. Since $L_N \geq N$ for all $N \geq 1$, we can assume without loss of generality that $L_N = N$. Consider the linear form

$$\psi(x_1, \dots, x_h) = u_1 x_1 + \dots + u_h x_h$$

Then

$$\varphi(a_1, \dots, a_h, b) = \psi(a_1, \dots, a_h) + v b$$

for all integers a_1, \dots, a_h, b . Moreover, since the sets A_1, \dots, A_h are finite, there is a positive integer g^* such that $\psi(\mathcal{A}) \subseteq [-g^*, g^*]$. If $(a_1, \dots, a_h) \in A_1 \times \dots \times A_h$, if $b \in \mathbf{Z}$, and if $\varphi(a_1, \dots, a_h, b) = n \in [-N, N]$, then

$$v|b| = |n - \psi(a_1, \dots, a_h)| \leq |n| + |\psi(a_1, \dots, a_h)| \leq N + g^*.$$

Replacing the set B_N with $B_N \cap [-(N + g^*)/v, (N + g^*)/v]$, we can assume without loss of generality that $B_N \subseteq [-(N + g^*)/v, (N + g^*)/v]$ for all $N \geq 1$.

We shall construct inductively an increasing sequence of finite sets $B'_1 \subseteq B'_2 \subseteq \dots$ with the following properties:

- (1) For every positive integer i and every integer $n \in [-i, i]$ we have $R_{\mathcal{A}, B'_i}^{(\varphi)}(n) = f(n)$.
- (2) For every positive integer i there is a strictly increasing sequence $\{N_j^{(i)}\}_{j=1}^\infty$ such that $i \leq N_1^{(i)}$ and $B'_i \subseteq B_{N_j^{(i)}}'$ for all $j \geq 1$.

We begin by constructing the set B'_1 . If $(a_1, \dots, a_h) \in A_1 \times \dots \times A_h$, if $b \in \mathbf{Z}$, and if $\varphi(a_1, \dots, a_h, b) \in [-1, 1]$, then

$$|b| \leq \frac{1 + g^*}{v}.$$

For all $N \geq 1$ we have $R_{\mathcal{A}, B_N}^{(\varphi)}(n) = f(n)$ for $|n| \leq N$, and so $R_{\mathcal{A}, B_N}^{(\varphi)}(n) = f(n)$ for $|n| \leq 1$. Let

$$B_N^{(1)} = B_N \cap \left[-\frac{1+g^*}{v}, \frac{1+g^*}{v} \right]$$

for $N \geq 1$. Then $\{B_N^{(1)}\}_{N=1}^\infty$ is an infinite sequence of subsets of the finite set $[-(1+g^*)/v, (1+g^*)/v] \cap \mathbf{Z}$. By the pigeonhole principle, there is a strictly increasing sequence $\{N_j^{(1)}\}_{j=1}^\infty$ of positive integers and a set B'_1 such that $1 \leq N_1^{(1)}$ and

$$B'_1 = B_{N_1^{(1)}}^{(1)} \subseteq B_{N_j^{(1)}}^{(1)}$$

for all $j \geq 1$.

Suppose that we have constructed an increasing sequence of sets $B'_1 \subseteq B'_2 \subseteq \dots \subseteq B'_i$ satisfying properties (1) and (2). For $j \geq 1$ we define the finite set

$$B_{N_j^{(i)}}^{(i+1)} = B_{N_j^{(i)}} \cap \left[-\frac{i+1+g^*}{v}, \frac{i+1+g^*}{v} \right].$$

Then $\{B_{N_j^{(i)}}^{(i+1)}\}_{j=1}^\infty$ is an infinite sequence of subsets of the finite set $[-(i+1+g^*)/v, (i+1+g^*)/v] \cap \mathbf{Z}$. By the pigeonhole principle, there is a strictly increasing sequence $\{N_j^{(i+1)}\}_{j=1}^\infty$ of positive integers and a set B'_{i+1} such that $i+1 \leq N_1^{(i+1)}$ and

$$B'_i \subseteq B'_{i+1} = B_{N_j^{(i+1)}}^{(i+1)} \subseteq B_{N_j^{(i+1)}}^{(i+1)}$$

for all $j \geq 1$. Properties (1) and (2) are satisfied for $i+1$. This completes the induction. Moreover, the set $A_h = \bigcup_{i=1}^\infty B'_i$ satisfies $R_{\mathcal{A}, A_h}^{(\varphi)}(n) = f(n)$ for all $n \in \mathbf{Z}$. This completes the proof. \square

Theorem 4. *Let $h \geq 1$ and $\varphi(x_1, \dots, x_h, y) = u_1x_1 + \dots + u_hx_h + y$. Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h -tuple of nonempty finite sets of integers and let $t \geq 1$. Suppose that there is a strictly increasing sequence $\{L_N\}_{N=1}^\infty$ of positive integers such that, for every $N \geq 1$, there exists a set B_N of integers and a set I_N consisting of $2L_N+1$ consecutive integers such that*

$$R_{\mathcal{A}, B_N}(n) = t \quad \text{for } n \in I_N.$$

Then there exists a set B such that

$$R_{\mathcal{A}, B}(n) = t \quad \text{for all } n \in \mathbf{Z}.$$

Proof. For every integer $N \geq 1$, there is an integer c_N such that $I_N = [c_N - L_N, c_N + L_N] \cap \mathbf{Z}$. Replace the set B_N with the set $B_N - c_N$ and apply Theorem 3. This completes the proof. \square

A related result appears in Nathanson [5].

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